The Value of Information in an (R,s,Q) Inventory Model

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Abstract

In this paper we compare three methods for the determination of the reorder point s in an (R, s, Q) inventory model subject to a service level constraint. The three methods differ in the modelling assumptions of the demand process which in turn leads to three different approximations for the distribution function of the demand during the lead time. The first model is most common in the literature, and assumes that the time axis is divided in time units (e.g. days). It is assumed that the demands per time unit are independent and identically distributed random variables. The second model monitors the customers individually. In this model it is assumed that the demand process is a compound renewal process, and that the distribution function of the interarrival times as well as that of the demand per customer are approximated by the first two moments of the associated random variable. The third method directly collects information about the demand during the lead time plus undershoot, avoiding convolutions of stochastic random variables and residual lifetime distributions. Consequently, the three methods require different types of information for the calculation of the reorder point in an operational setting. The purpose of this paper is to derive insights into the value of information; therefore it compares the target service level with the actual service level associated with the calculated reorder point. It will be shown that the performance of the first model (discrete time model) depends on the coefficient of variation of the interarrival times. Furthermore, because we use asymptotic relations in the compound renewal model, we derive some bounds for the input parameters within which this model applies. Finally we show that the aggregated information model is superior to the other two models.

1. Introduction

The (R, s, Q) inventory model is well-known in the literature, and is frequently used in practice. In this paper a comparison is made of three methods for determining the minimal value for the reorder point s such that a target value of the P_2 customer service level is guaranteed, where the P_2 service level is defined as the fraction of demand directly delivered from stock on hand. Under the regime of the (R, s, Q) inventory policy the inventory position is monitored every R time units in order to take a replenishment decision. When the inventory position is below s, an amount of Q units is ordered such that the inventory position is raised to a value between s and $s + Q$. Customer orders which cannot be delivered directly from stock, will be backordered.

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Both the supplier and the retailer should have certain advantages when using an (R,s,Q) inventory model instead of another replenishment policy. Firstly, using a periodic review replenishment policy (with a sufficiently large review period) for products replenished by the same supplier, the ordering and transportation costs can be reduced when replenishment orders can be properly coordinated. Secondly, for make-to-order organizations the knowledge of review moments of its customers (which are possible demand epochs), and the fact that the customers request fixed quantities, can be translated into an efficient production schedule. This clearly reduces the production lead times, and consequently has a positive effect on the required inventory at the retailer, needed to achieve a desired customer service level. Thus, the retailer benifits due to coordination and shorter lead times, whereas the manufacturer benifits due to the efficient production schedules and less work in progress. Furthermore, the (R, s, Q) inventory model coincides with the time phased reorder point in a MRP-system (Material Requirement Planning).

To use mathematical models such as inventory or production planning models in practice, information about the underlying processes is required. For the above mentioned inventory model often information is available or collected about demand per time unit. However, more accurate information could be collected by monitoring the customers individually, thereby collecting information about the interarrival times and the demand size per customer. An interesting question would be: what is the additional value of this more detailed information? In this paper three different ways to model the demand process are described, and therefore three different ways of collecting information about the actual demand and lead time process are required. In the sequel we denote these methods as the discrete time method (DTM), the compound renewal method (CRM), and the aggregated information method (AIM). The DTM is well-known in the literature; the CRM and the AIM, however, are not common practice. The CRM is presented in De Kok (1991b); Sahin (1983) derived expression for the state probabilities of the inventory position in the (s, S) inventory model with compound renewal demand. The idea of measuring aggregated information (the main idea behind AIM) is described in, for example, Brown (1963), and Strijbosch and Heuts (1992).

The DTM is extensively described in the literature (see, for example, Schneider (1981,1990), Tijms and Groenevelt (1984), Silver and Peterson (1985), and Tersine (1994)). The DTM approach assumes that the time axis is divided into disjunct time units of length T (e.g. days). Moreover, it is assumed that information is available about the first two moments of the demand per time unit (obtained from historical data). To reduce the complexity of the model, it is assumed that the demands per time unit are independent and identically distributed random variables (in general this random variable might have positive probability mass at zero). Notice that the setting in which R is equal to T is often denoted as an (s, Q) or (Q, r) inventory model. A method which closely resembles the DTM, is the method described by Dunsmuir and Snyder (1989) and Janssen et al. (1996), where the demand is modelled as a compound Bernoulli process, that is, with a fixed probability there is positive demand during a time unit, else demand is zero.

In the CRM the time axis is *not* divided into disjunct intervals. The demand process can be described as a compound renewal process, which is obviously a generalization of a compound Poisson process; see also Sahin (1983, 1989). In practice, customers are often retailers who control their inventory with, for example, (s, Q) policies. In these situations the demand process, which is the superposition of the ordering processes of the retailers, is certainly not a compound Poisson process. For CRM it is assumed that information is available about the first two moments of the interarrival times of customers as well as about the first two moments of the demand sizes of each customer. It is clear that the CRM requires specific and more detailed customer information, which is often not available in practice, whereas the DTM requires only information about demands per time unit.

Both methods require convolutions of stochastic random variables and residual lifetime distributions. The AIM monitors undershoots plus demands during subsequent lead times, it is assumed that this stochastic variable forms a sequence of independent and identically distributed random variables. By directly observing this aggregated quantity AIM avoids the calculation of convolutions of stochastic random variables and moments of residual lifetime distributions. Finally, the distribution function of the undershoot plus the lead time demand is fitted, using the estimated moments of the historical data. The main purpose of the paper is to compare the required service level with the actual service associated to the reorder level calculated by each of the methods, and give an answer to the question stated above: what is the benifit of using more or less detailed information with respect to the quality of reorder point calculation with the associated method. To cover inventory practice as closely as possible, we suppose that the actual underlying demand process is a compound renewal process. Notice that in this way we neglect possible autocorrelations in the interarrival times and demand sizes. The impact of correlated demand during the lead time is shown by e.g. Ray (1980,1981) and Fotopoulos et al. (1988).

The organisation of the paper is as follows. In section 2 the general assumptions of the (R, s, Q)- inventory model are presented, while in sections 3, 4, and 5 the AIM, DTM, and CRM respectively are described. In section 6 several numerical comparisons are presented, based on discrete event simulation. Finally, in section 7 conclusions are given, and future research is discussed.

2. The (R, s, Q) model description

In order to specify the inventory model we distinguish between the demand process and the lead time process. We assume that the demand process is a compound renewal process, i.e. the interarrival times $A_2, A_3, \ldots$ and the demands per customer $D_1, D_2, \ldots$ are independent and identically distributed random variables. The interarrival times of customers are independent of the demands per customer. Moreover, we assume that the process is already going on for an infinite period of time and that time epoch zero, which is a review moment, is an arbitrary point in time, indicating that A_1 is distributed according to a residual lifetime distribution. The reasons for this assumption will be explained later in this paper. We assume that the lead times $L_1, L_2, \ldots$ do not cross in time, implying that the lead times of replenishment orders are dependent random variables. The first moment, standard deviation and coefficient of variation of a generic random variable X will be denoted respectively by $\mathbb{E}X, \sigma(X)$ and c_X.

The main problem is to determine the minimal value of the reorder level s, given values for Q and R, such that a target service level is achieved. Hence, we assume that R and Q are determined based on cost considerations, such as a trade-off between the replenishment costs and the inventory costs. We derive a general relation between the reorder level s and the target service level $\beta(R, s, Q)$ for the (R, s, Q) inventory model. We define the following variables:

$N(t_1, t_2)$	:=	the number of customer arrivals in $(t_1, t_2]$;
$Z(t_1, t_2)$	:=	the total demand in $(t_1, t_2]$;
$Z_{t,i}$	:=	$Z(it, (i+1)t)$ where $t \in \mathbb{R}^+$ and $i \in \mathbb{N}$;
$X(t)$	:=	the inventory position at time t just before a replenishment order is placed;
T_k	:=	the point in time at which the inventory position drops below s the k-th time after 0;
U_k	:=	$s - X(T_k)$, the undershoot at time T_k;
τ_k	:=	the first review moment after T_k;
$U_{R,k}$	:=	$s - X(\tau_k)$, the undershoot at the review epoch;
Z_k	:=	$Z(\tau_k, \tau_k + L_k) + U_{R,k}$;

We now focus on the first replenishment cycle starting after time 0, and restrict ourself to the situations where all replenishments are equal to Q. Thus the undershoot is always smaller than Q, of course this only holds when the demand during the review period $Z_{R,i}$ is much smaller that Q. Because all the processes involved are stationary we conclude that $Z(\tau_1, \tau_1 + L_1) \stackrel{d}{=} Z(\tau_2, \tau_2 + L_2)$ and $U_{R,1} \stackrel{d}{=} U_{R,2}$, where $\stackrel{d}{=}$ denotes equality in distribution. Given that the backlog at the end of the cycle equals the backlog at the beginning of the cycle together with the unsatisfied demand during the cycle, it can be derived (see e.g. de Kok(1991b)) that s must satisfy the following service equation (see also Figure 1):

$$\beta(R, s, Q) = \begin{cases} 0 & s \leq -Q \\ 1 - \frac{\boldsymbol{E} Z_1 - s - \boldsymbol{E}(Z_1 - s - Q)^+}{Q} & -Q < s \leq 0 \\ 1 - \frac{\boldsymbol{E}(Z_1 - s)^+ - \boldsymbol{E}(Z_1 - s - Q)^+}{Q} & 0 < s \end{cases} \tag{1}$$

where $X^+ = \text{MAX}(0, X)$.

From (1) the reorder point s can be calculated according to the method presented by Tijms and Groenevelt (1984), where in order to calculate $\mathbb{E}(Z_1 - s)^+$ and $\mathbb{E}(Z_1 - s - Q)^+$ it is assumed that the distribution of Z_1 can be approximated by a generalized Erlang distribution using the first two moments of Z_1. Hence, to calculate the reorder point s only the first two moments of Z_1 are required. Basically, the AIM, DTM and the CRM differ in the way these moments are obtained.

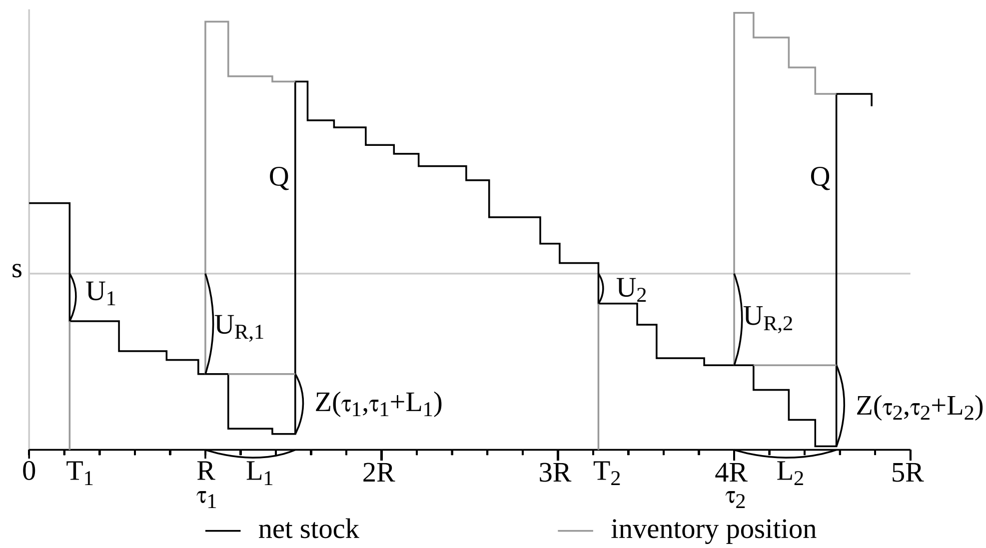

Figure 1: Evolution of the net stock and inventory position during the first replenishment cycle

3. The Aggregated Information Method (AIM)

The AIM is based on the estimates of the first two moments of Z_1, which are obtained by collecting information from historical data of this quantity. To be more precise, each time the inventory position drops below s, and consequently a replenishment order is placed, the actual undershoot has to be registered, and furthermore, the accumulated demands until the replenishment order arrives have to be registered. The sum of these two variables is a realisation of the variable representing Z_1.

We notice that the distribution function of the undershoot is asymptotically invariant under changes of the reorder point s. However, this is not true for the reorder quantity Q and the review period R. Hence, the moments for the undershoot must be obtained from historical data under a fixed reorder quantity and review period.

The main advantage of the AIM is that moments of convolutions and moments of the residual lifetime distribution need not to be calculated. Furthermore, correlations between demands or between demands and lead times are allowed. These effects are all included by directly collecting information about Z_1. The presence of correlations is often quite cumbersome when applying theoretical inventory models in real life situations, as well in forecasting procedures.

However, a possible disadvantage of the AIM (which has to be investigated in future research) is the small number of historical data. The aggregated variable is composed of several components, and therefore it is hard to estimate this variable. In order to make fair comparisons the exact knowledge of the required information is pre-assumed. This of course is seldomly the case in practice.

4. The Discrete Time Method (DTM)

In this section we describe the DTM, for which we assume that the time axis is divided into time units of equal length T, e.g. days, and that R and the lead times L_k for $k = 1, 2, \dots$ are integral numbers of T. Furthermore, we assume that the inventory position is monitored every time unit. Decisions about replenishments, however, are made every R time units. The depletion of the inventory position in the k-th time unit is equal to $Z_{T,k}$, with $\mathbb{P}(Z_{T,k} = 0) > 0$. To obtain tractable results for the first two moments of Z_1 we have to assume that the $Z_{T,k}$'s are independent random variables, which is not necessarily true in practice. We note here that this assumption is often made in practice and most text books without checking its validity.

Due to the transformation of the continuous time axis to a slotted time axis, events of several types may coincide in time with positive probability. Therefore, we have to specify the priority rule in which sequence the events are handled. Notice that different priority rules lead to a different value for the reorder point. We assume that the depletions $Z_{T,k}$ are handled before replenishment orders at the end of a time unit. This reflects the situation where stock depletions during a time unit are accumulated until the end of the time unit, just before the net stock and inventory position are adjusted, and subsequently the arrivals of replenishment orders are handled. In Figure 2 the same sample path is used as in Figure 1, to illustrate the

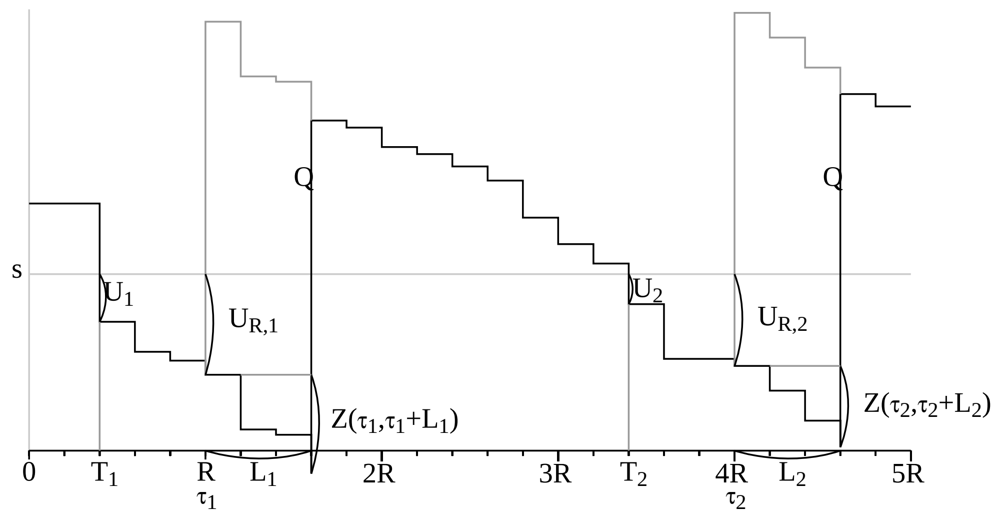

Figure 2: Evolution of the net stock and inventory position during the first replenishment cycle

transformation to a lattice time basis and the impact of the priority rule. This priority rule, as can be seen in Figure 2, leads to a service which is lower than the service when measured continuously. Under this priority rule the demand during the lead time is given by

$$Z(\tau_1, \tau_1 + L_1) = \sum_{j=\tau_1+1}^{\tau_1+L_1} Z_{T,j} \tag{2}$$

Because the $Z_{T,j}$'s are non-negative i.i.d. random variables we can apply a well-known result for the first two moments of a stochastic number of i.i.d. random variables:

$$\begin{aligned} \mathbb{E}Z(\tau_1, \tau_1 + L_1) &= \mathbb{E}L_1 \mathbb{E}Z_{T,1} && (3) \\ \mathbb{E}Z(\tau_1, \tau_1 + L_1)^2 &= \mathbb{E}L_1 \sigma^2(Z_{T,1}) + \mathbb{E}L_1^2 (\mathbb{E}Z_{T,1})^2 && (4) \end{aligned}$$

Furthermore we have

$$Z_{R,0} = \sum_{j=1}^{R} Z_{T,j} \tag{5}$$

and hence

$$\begin{aligned} \mathbb{E}Z_{R,0} &= R\mathbb{E}Z_{T,1} && (6) \\ \mathbb{E}Z_{R,0}^2 &= R\sigma^2(Z_{T,1}) + R^2(\mathbb{E}Z_{T,1})^2 && (7) \end{aligned}$$

When, for example, we assume that $Z_{R,0}$ is gamma distributed the third moment of $Z_{R,0}$ is given by

$$\mathbb{E}Z_{R,0}^3 = (1 + c_{Z_{R,0}}^2)(1 + 2c_{Z_{R,0}}^2)(\mathbb{E}Z_{R,0})^3 \tag{8}$$

Because the $Z_{T,k}$'s are i.i.d. and the $Z_{R,i}$'s are disjunct collections of an identical number of $Z_{T,k}$'s, it can be concluded that also the $Z_{R,i}$'s are i.i.d. The distribution function of $U_{R,1}$ can be approximated by the asymptotic residual lifetime distribution of the renewal process generated by the sequence $(Z_{R,0}, Z_{R,1}, \ldots)$. Using well-known results from renewal theory yields (see Tijms (1994), pp. 10):

$$\mathbb{E}U_{R,1} \simeq \frac{\mathbb{E}Z_{R,0}^2}{2\mathbb{E}Z_{R,0}} \tag{9}$$

$$\mathbb{E}U_{R,1}^2 \simeq \frac{\mathbb{E}Z_{R,0}^3}{3\mathbb{E}Z_{R,0}} \tag{10}$$

It is known from numerical investigations that for practical purposes the relations (9) and (10) hold when $Q \geq t_0$, (see Tijms (1994), pp.14) where

$$t_0 = \begin{cases} \frac{3}{2}c_{Z_{R,0}}^2 \mathbb{E}Z_{R,0} & \text{if} \quad c_{Z_{R,0}}^2 > 1 \\ \mathbb{E}Z_{R,0} & \text{if} \quad 0.2 < c_{Z_{R,0}}^2 \leq 1 \\ \frac{1}{c_{Z_{R,0}}}\mathbb{E}Z_{R,0} & \text{if} \quad 0 < c_{Z_{R,0}}^2 \leq 0.2 \end{cases} \tag{11}$$

Substition of the two moments of $Z_{T,1}$ and the first two moments of L_1 (which can be obtained from historical data) in (3),(4) and (6) to (10) enables us to calculate the first two moments of Z_1. Next the distribution function of Z_1 is approximated by a generalized Erlang distribution (see e.g. Tijms and Groenevelt (1984)) and relation (1) is used to compute the reorder point s.

Remark 4.1 : Note that Z_k for $k = 1, 2, \ldots$ also can be written as the undershoot U_k under s at T_k plus the demand during the pseudo lead time ($\hat{L}_k := L_k + \tau_k - T_k$). However, it can be proven that both approaches result in the same expressions for the first two moments of Z_k.

5. The Compound Renewal Method (CRM)

In this situation we assume that the demand process is modelled as a compound renewal process $(A_1, D_1), (A_2, D_2), \ldots$. We again assume that the process is already going on for an infinite time period. Because zero is an arbitrary point in time, it can be concluded that A_1 represents a residual interarrival time, whereas A_2 denotes a generic interarrival time. In contrast with the previous two sections we don't focus on demand during a time unit but on demand per customer.

The approach is again based on the application of formula (1), where we assume that the distribution of Z_1 can be approximated by that of a generalized Erlang distribution, which implies that for application of (1) we only need the first two moments of Z_1, where Z_1 was defined as $Z_1 = Z(\tau_1, \tau_1 + L_1) + U_{R,1}$.

In order to obtain expressions for $U_{R,1}$, the same approach as described in the DTM is followed. Analoguously to the derivation of relation (6) and (7), we find

$$\mathbb{E} Z_{R,0} = \mathbb{E} N(0,R) \mathbb{E} D_1 \tag{12}$$

$$\mathbb{E} Z_{R,0}^2 = \mathbb{E} N(0,R) \sigma^2(D_1) + \mathbb{E} N(0,R)^2 (\mathbb{E} D_1)^2 \tag{13}$$

where the moments of $N(0,R)$ can be approximated by asymptotic relations derived with renewal theory (for the first two moments see Tijms(1994), whereas the third moment can be obtained in a similar way)

$$\mathbb{E} N(0,R) \simeq \frac{R}{\alpha_1} \tag{14}$$

$$\mathbb{E} N(0,R)^2 \simeq \frac{R^2}{\alpha_1^2} + R\left(\frac{\alpha_2}{\alpha_1^3} - \frac{1}{\alpha_1}\right) + \frac{\alpha_2^2}{2\alpha_1^4} - \frac{\alpha_3}{3\alpha_1^3} \tag{15}$$

where $\alpha_k := \mathbb{E} A_2^k$.

These asymptotic relations are valid when $R \geq t_1$ (see Tijms(1994), pp 14) where

$$t_1 = \begin{cases} \frac{3}{2} c_{A_2}^2 \mathbb{E} A_2 & \text{if } \; c_{A_2}^2 > 1 \\ \mathbb{E} A_2 & \text{if } \; 0.2 < c_{A_2}^2 \leq 1 \\ \frac{1}{c_{A_2}} \mathbb{E} A_2 & \text{if } \; 0 < c_{A_2}^2 \leq 0.2 \end{cases} \tag{16}$$

Notice that relation (15) contains the third moment of A_2. However, the estimates of higher moments (third and higher) of a stochastic variable are very sensitive for extreme values in the data. Therefore, we propose to approximate the distribution function of A_2 by a gamma distribution in order to calculate $\mathbb{E} A_2^3$ based on the first two moments of A_2. Then combining relations (12) and (13) with the asymptotic relations (14) and (15) yields

$$\mathbb{E}Z_{R,0} \simeq \frac{R}{\alpha_1}\mathbb{E}D_1 \tag{17}$$

$$\mathbb{E}Z_{R,0}^2 \simeq \Big(\frac{R^2}{\alpha_1^2} + \frac{R}{\alpha_1}(c_{A_2}^2 + c_{D_1}^2) + \tfrac{1}{6}(1 - c_{A_2}^4)\Big)\mathbb{E}D_1^2 \tag{18}$$

Using that $\sigma(Z_{R,0}) \geq 0$, it is easy to see that (18) is only valid when:

$$c_{A_2}^2 \in [0, \frac{3R}{\alpha_1} + \sqrt{\frac{9R^2}{\alpha_1^2} + \frac{6R}{\alpha_1}c_{D_1}^2 + 1}). \tag{19}$$

Hence, when $\mathbb{E}A_2$ is large with respect to R the region of application is restricted through the condition (19) for $c_{A_2}^2$. However, if $R \ll \mathbb{E}A_2$, the frequency with which the inventory position is monitored (in order to make a replenishment decision) when the (R,s,Q) inventory model is applied is larger than the frequency of customer arrivals in that inventory system, and therefore it is evident to use a continuous review inventoy model, such as the (s, Q) or (s, S) inventory model.

Now, using the fact that the distribution of the undershoot has approximately a residual lifetime distribution and applying results from renewal theory, we find

$$\mathbb{E}U_{R,1} \simeq \frac{\mathbb{E}Z_{R,0}^2}{2\mathbb{E}Z_{R,0}} \tag{20}$$

$$\mathbb{E}U_{R,1}^2 \simeq \frac{\mathbb{E}Z_{R,0}^3}{3\mathbb{E}Z_{R,0}} \tag{21}$$

which can be calculated by again using (8) in addition to (17) and (18). In order to derive expressions for the first two moments of Z_1, in addition to (20) and (21) we also need expressions for the first two moments of de demand during the lead time. Again, it is easily seen that

$$\mathbb{E}Z(0, L) = \mathbb{E}N(0, L)\mathbb{E}D_1 \tag{22}$$

$$\mathbb{E}Z(0, L)^2 = \mathbb{E}N(0, L)\sigma^2(D_1) + \mathbb{E}N(0, L)^2(\mathbb{E}D_1)^2 \tag{23}$$

where the moments of $N(0, L)$ can be approximated by its asymptotic relations

$$\mathbb{E}N(0, L) \simeq \frac{\mathbb{E}L_1}{\alpha_1} \tag{24}$$

$$\mathbb{E}N(0, L)^2 \simeq \frac{\mathbb{E}L_1^2}{\alpha_1^2} + \mathbb{E}L_1\left(\frac{\alpha_2}{\alpha_1^3} - \frac{1}{\alpha_1}\right) + \frac{\alpha_2^2}{2\alpha_1^4} - \frac{\alpha_3}{3\alpha_1^3} \tag{25}$$

which only hold when $I\!\!E L \geq t_1$ see (16). For the special case that A_2 is gamma distributed the following simple approximate relations can be derived

$$I\!\!E Z(0, L_1) \simeq \frac{I\!\!E L_1}{\alpha_1} I\!\!E D_1 \tag{26}$$

$$I\!\!E Z(0, L_2)^2 \simeq \Big(\frac{I\!\!E L_1^2}{\alpha_1^2} + \frac{I\!\!E L_1}{\alpha_1}(c_{A_2}^2 + c_{D_1}^2) + \tfrac{1}{6}(1 - c_{A_2}^4)\Big)(I\!\!E D_1)^2 \tag{27}$$

Analoguously to condition (19) for (18) we derive the following condition for the validity of (27):

$$c_{A_2}^2 \in [0, \frac{3 I\!\!E L_1}{\alpha_1} + \sqrt{\frac{9(I\!\!E L_1)^2}{\alpha_1^2} + \frac{6\sigma(L_1)^2}{\alpha_1^2} + \frac{6 I\!\!E L_1}{\alpha_1} c_{D_1}^2 + 1}). \tag{28}$$

For example, when $\sigma_L^2 = 0$ and $c_D^2 = 1$ it is easy to see that $c_{A_2}^2 \in [0, 6\frac{\boldsymbol{E}L}{\boldsymbol{E}A_2} + 1)$. Hence, when the number of customer arrivals during the lead time goes to zero the maximal value for $c_{A_2}^2$ is equal to one.

Thus using (12) to (18) and (20) to (25) we can find expressions for the first two moments of Z_1, which enables us to calculate the reorder point s.

6. Simulation experiments

In the experiments we consider two criteria for comparison, namely the reorder point s calculated for each method and the associated actual service level. We use a compound renewal process to model the demand process. More specifically the interarrival times and the demand size of a customer are independent and identically distributed (i.i.d.) random variables with generalized Erlang distributions. Moreover, the lead times are also i.i.d. random variables with a generalized Erlang distribution function. Thus, in order to describe the inventory model, we have to specify values for the first two moments of the interarrival times, demand sizes of customers and lead times, the length of the review period (R), the replenishment quantity (Q), and the target service level (β). In table 6.1 the parameters of our experiments are given.

Table 6.1: Input parameters for simulation experiments

Q	β	R	$\mathbb{E}A_2$	$\sigma(A_2)$	$(\mathbb{E}L_1, \sigma(L_1))$	$\mathbb{E}D_1$	$\sigma(D_1)$
50,100	0.90, 0.95	5	0.5,1,2, 10	0.25, 0.5, 1, 1.5, 2, 3	(4,0),(10,2)	5	5

To obtain sufficiently accurate values for the relevant input variables (e.g. the first two moments of the demand per time unit) for the various methods, all the required input values are derived from a preceding simulation run. In operational settings this coincides with exact knowledge about the moments of the relevant random variables. It is well-known that small sample sizes can lead to bad estimates, especially for the second moment of a random variable; see e.g. Silver and Peterson (1985).

For each of the three methods a simulation is performed with the same seeds for the pseudo random generator, in order to sample the required random variables. We first simulated 1 subrun of 100,000 time units to obtain reliable values for moments of the relevant random variables. Secondly, the reorder point s is calculated by the associated method. We simulated 10 times 100,000 time units and calculated the 95% confidence interval for the actual service level achieved in the simulation experiments. The results are tabulated in Appendix 1.

Basically three major conclusion can be drawn from the simulation experiments (which are explained below), namely

- the performance of the DTM is bad for situations with $c_{A_2} \neq 1$;
- the performance of the CRM is good for most situations, except when Q is too small or $\mathbb{E}A_2$ is too large;
- the AIM is superior to CBM and DTM.

The performance of the DTM depends heavily on the coefficient of variation of the interarrival times. When the interarrival times are almost constant ($c_{A_2} < 1$), the DTM tends to overestimate the reorder point s; when the interarrival times are erratic ($c_{A_2} > 1$), the DTM tends to underestimate the reorder point; see Figures 3 and 4. The explanation for this behaviour is that in situations where $c_{A_2} \neq 1$ the independency assumption of the D_n is violated. Only when $c_{A_2} = 1$, which represent the compound Poisson process, the D_n are

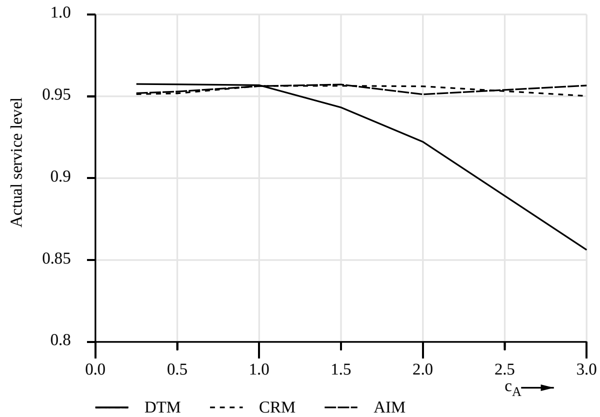

Figure 3: Actual service levels in case $(\mathbb{E}L, \sigma(L)) = (10, 2)$, $\mathbb{E}A_2 = 2$, $Q = 50$

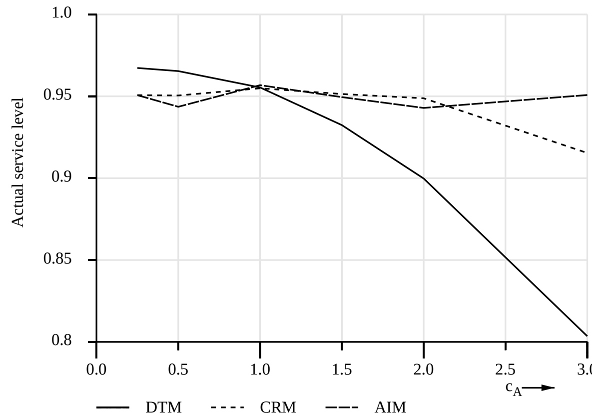

Figure 4: Actual service levels in case $(\mathbb{E}L, \sigma(L)) = (4, 0)$, $\mathbb{E}A_2 = 1.0$, $Q = 50$

i.i.d. This means that the DTM is valid only when the demand process is a compound Poisson case. In situation with $c_{A_2} \neq 1$ the DTM is not valid; thus even in those situations where the moments of the demand and lead time processes are known exactly, the method performs badly.

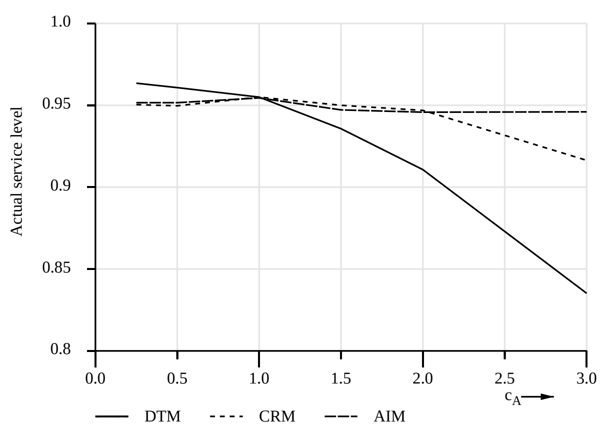

Figure 5: Actual service levels in case $(\mathbb{E}L, \sigma(L)) = (10, 2)$,$\mathbb{E}A_2 = 2$, $Q = 100$

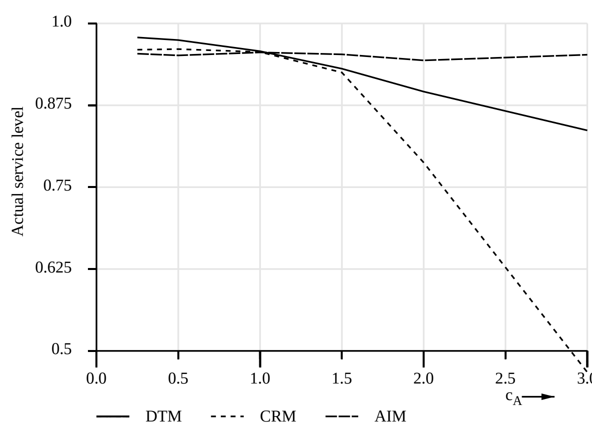

Figure 6: Actual service levels in case $(\mathbb{E}L, \sigma(L)) = (10, 2)$, $\mathbb{E}A_2 = 10$, $Q = 50$

The second conclusion is that the CRM performs excellently in almost every situation. However, in situations where the interarrival times are very erratic (e.g. $\sigma(A_2) = 4, \mathbb{E}A_2 =$

0.5) and the lead times are relatively short, this method performs badly. This can be explained by the well-known fact (see Tijms (1994) pag. 14) that the asymptotic approximations for the moments of the number of renewals in a short time interval are of poor quality (for a good bound see expression (11)). Furthermore, in situations where Q is relatively small compared to the expected demand size per customer, the CRM not always performs satisfactorily. When $\mathbb{E}A_2$ is large with repect to R, the CRM also does not perform very well, but as argued before the continuous (s, Q) inventory model should be considered in that case.

Finally, we see that AIM is superior to the others. All complications of determining the complex stochastic quantities U_1 and Z_1 are avoided, simply by collecting information about these measurable quantities. As has been noted before, we use exact knowledge about these quantities, which requires many occurrences of those quantities. In practice this information is often not available.

7. Conclusions and future research

In this paper we compared three methods for the determination of the reorder point s in an (R, s, Q) inventory model subject to a service level constraint. The three methods differ in the modelling assumptions for the demand process, and therefore require different levels of information to feed the inventory models. We compared the quality of the methods by discrete event simulation, and conclude that each method is applicable only in a restricted area. Furthermore, it turns out that the AIM is applicable in every situation. However, in using this method the problem of determining moments of complex stochastic variables is shifted to the problem of properly estimating these complex stochastic variables. The choice of method to use in practical situations should be based on the quality of the information available. Hence, when good estimates are available for the demand during the lead time plus undershoot, it is evident to aim at AIM. If the quality of the aggregated variables is doubtful, more disaggregated approaches such as the DTM and CRM should be considered. The choice must be based on the coefficient of variation of the interarrival times.

For future reseach we would like to point out two extensions. Firstly, when using these models in practice, estimates for the moments of the relevant stochastic variables are inevitable. Hence, the impact of the quality of the estimates on the performance of the methods should be investigated; see also Vaughan (1995). Secondly, instead of using estimates for the moments of the variables, one could also integrate forecasting procedures (e.g. exponential smoothing methods) directly in inventory models.

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Appendix 1: Results of numerical experiments

Table A.1. $\left(R = 5, (I\!\!E D\,;\,\sigma_D) = (5\,;\,5)), (I\!\!E L\,;\,\sigma_L) = (10\,;\,2), \beta = 0.95\right)$

		DTM		CRM		AIM	
Q	$(I\!\!E A, c_A)$	s	$\hat{\beta}$	s	$\hat{\beta}$	s	$\hat{\beta}$
50	(0.50 , 0.25)	173.4	0.9607 (± 0.0034)	170.8	0.9562 (± 0.0043)	173.4	0.9617 (± 0.0039)
50	(0.50 , 0.50)	176.2	0.9613 (± 0.0029)	174.3	0.9558 (± 0.0038)	176.1	0.9606 (± 0.0024)
50	(0.50 , 1.00)	189.1	0.9626 (± 0.0028)	187.1	0.9578 (± 0.0025)	190.0	0.9626 (± 0.0027)
50	(0.50 , 1.50)	202.6	0.9536 (± 0.0033)	206.1	0.9591 (± 0.0019)	208.4	0.9612 (± 0.0031)
50	(0.50 , 2.00)	215.9	0.9394 (± 0.0022)	229.6	0.9562 (± 0.0025)	232.0	0.9588 (± 0.0038)
50	(0.50 , 3.00)	233.6	0.8899 (± 0.0019)	283.8	0.9539 (± 0.0016)	284.0	0.9544 (± 0.0016)
50	(1.00 , 0.25)	89.2	0.9591 (± 0.0037)	86.7	0.9515 (± 0.0045)	87.6	0.9554 (± 0.0047)
50	(1.00 , 0.50)	91.9	0.9599 (± 0.0018)	89.5	0.9517 (± 0.0020)	89.2	0.9513 (± 0.0038)
50	(1.00 , 1.00)	99.9	0.9550 (± 0.0053)	99.8	0.9556 (± 0.0044)	98.2	0.9513 (± 0.0067)
50	(1.00 , 1.50)	108.8	0.9419 (± 0.0055)	115.0	0.9550 (± 0.0053)	113.6	0.9539 (± 0.0065)
50	(1.00 , 2.00)	116.0	0.9179 (± 0.0050)	133.0	0.9542 (± 0.0040)	128.7	0.9465 (± 0.0035)
50	(1.00 , 3.00)	124.0	0.8514 (± 0.0033)	170.0	0.9507 (± 0.0025)	173.0	0.9540 (± 0.0019)
50	(2.00 , 0.25)	49.5	0.9673 (± 0.0020)	44.5	0.9507 (± 0.0029)	44.7	0.9507 (± 0.0035)
50	(2.00 , 0.50)	50.6	0.9654 (± 0.0030)	46.7	0.9505 (± 0.0031)	45.4	0.9436 (± 0.0041)
50	(2.00 , 1.00)	54.6	0.9554 (± 0.0076)	54.5	0.9549 (± 0.0075)	55.1	0.9568 (± 0.0063)
50	(2.00 , 1.50)	59.7	0.9324 (± 0.0049)	65.8	0.9514 (± 0.0052)	64.6	0.9495 (± 0.0041)
50	(2.00 , 2.00)	62.7	0.8998 (± 0.0080)	78.0	0.9488 (± 0.0037)	76.6	0.9429 (± 0.0059)
50	(2.00 , 3.00)	65.2	0.8035 (± 0.0041)	92.6	0.9154 (± 0.0033)	107.0	0.9508 (± 0.0028)
50	(10.00 , 0.25)	15.2	0.9786 (± 0.0012)	11.1	0.9599 (± 0.0021)	10.0	0.9538 (± 0.0017)
50	(10.00 , 0.50)	15.3	0.9746 (± 0.0017)	12.2	0.9608 (± 0.0020)	10.8	0.9512 (± 0.0022)
50	(10.00 , 1.00)	15.9	0.9576 (± 0.0022)	15.6	0.9564 (± 0.0022)	15.5	0.9558 (± 0.0022)
50	(10.00 , 1.50)	16.8	0.9309 (± 0.0031)	15.9	0.9251 (± 0.0031)	20.8	0.9528 (± 0.0017)
50	(10.00 , 2.00)	17.2	0.8959 (± 0.0047)	7.2	0.7876 (± 0.0038)	25.0	0.9436 (± 0.0030)
50	(10.00 , 3.00)	17.3	0.8367 (± 0.0052)	-6.1	0.4675 (± 0.0044)	33.7	0.9522 (± 0.0034)
100	(0.50 , 0.25)	158.7	0.9582 (± 0.0031)	156.3	0.9531 (± 0.0032)	157.7	0.9548 (± 0.0031)
100	(0.50 , 0.50)	161.1	0.9564 (± 0.0035)	159.3	0.9537 (± 0.0040)	161.2	0.9557 (± 0.0033)
100	(0.50 , 1.00)	172.8	0.9590 (± 0.0050)	170.8	0.9556 (± 0.0047)	169.6	0.9543 (± 0.0038)
100	(0.50 , 1.50)	185.2	0.9524 (± 0.0028)	188.5	0.9551 (± 0.0042)	187.9	0.9539 (± 0.0039)
100	(0.50 , 2.00)	197.7	0.9408 (± 0.0033)	210.7	0.9564 (± 0.0038)	206.8	0.9530 (± 0.0041)
100	(0.50 , 3.00)	214.5	0.8933 (± 0.0022)	263.2	0.9541 (± 0.0019)	260.1	0.9511 (± 0.0023)
100	(1.00 , 0.25)	77.7	0.9560 (± 0.0019)	75.7	0.9503 (± 0.0023)	75.7	0.9502 (± 0.0023)
100	(1.00 , 0.50)	79.9	0.9554 (± 0.0032)	77.9	0.9508 (± 0.0029)	77.5	0.9495 (± 0.0032)
100	(1.00 , 1.00)	86.6	0.9521 (± 0.0029)	86.5	0.9517 (± 0.0028)	85.5	0.9497 (± 0.0025)
100	(1.00 , 1.50)	94.2	0.9418 (± 0.0031)	99.7	0.9540 (± 0.0043)	98.4	0.9506 (± 0.0040)
100	(1.00 , 2.00)	100.5	0.9203 (± 0.0047)	116.0	0.9530 (± 0.0032)	113.6	0.9491 (± 0.0028)
100	(1.00 , 3.00)	107.7	0.8620 (± 0.0036)	150.7	0.9499 (± 0.0020)	150.6	0.9495 (± 0.0022)
100	(2.00 , 0.25)	39.8	0.9635 (± 0.0029)	36.2	0.9504 (± 0.0030)	36.5	0.9516 (± 0.0028)
100	(2.00 , 0.50)	40.7	0.9608 (± 0.0036)	37.8	0.9497 (± 0.0048)	38.2	0.9516 (± 0.0042)
100	(2.00 , 1.00)	43.8	0.9550 (± 0.0055)	43.8	0.9550 (± 0.0055)	43.7	0.9545 (± 0.0057)
100	(2.00 , 1.50)	47.8	0.9357 (± 0.0063)	52.9	0.9500 (± 0.0041)	52.1	0.9472 (± 0.0038)
100	(2.00 , 2.00)	50.3	0.9107 (± 0.0071)	63.3	0.9469 (± 0.0043)	62.7	0.9458 (± 0.0045)
100	(2.00 , 3.00)	52.3	0.8352 (± 0.0055)	76.0	0.9164 (± 0.0028)	89.5	0.9460 (± 0.0017)

Table A.2. $\left(R = 5, (\mathbb{E}D\,;\,\sigma_D) = (5\,;\,5)\right), (\mathbb{E}L\,;\,\sigma_L) = (4\,;\,0), \beta = 0.95\Big)$

		DTM		CRM		AIM	
Q	$(\mathbb{E}A, c_A)$	s	$\hat{\beta}$	s	$\hat{\beta}$	s	$\hat{\beta}$
50	(0.50 , 0.25)	93.3	0.9577 (± 0.0017)	91.7	0.9530 (± 0.0018)	95.6	0.9641 (± 0.0024)
50	(0.50 , 0.50)	95.8	0.9573 (± 0.0018)	94.4	0.9528 (± 0.0026)	97.2	0.9610 (± 0.0015)
50	(0.50 , 1.00)	106.0	0.9591 (± 0.0020)	104.7	0.9566 (± 0.0018)	107.8	0.9630 (± 0.0021)
50	(0.50 , 1.50)	117.5	0.9532 (± 0.0033)	120.2	0.9580 (± 0.0029)	124.1	0.9644 (± 0.0030)
50	(0.50 , 2.00)	128.8	0.9373 (± 0.0024)	139.1	0.9559 (± 0.0039)	140.7	0.9582 (± 0.0034)
50	(0.50 , 3.00)	144.6	0.8936 (± 0.0020)	182.2	0.9560 (± 0.0015)	181.9	0.9556 (± 0.0015)
50	(1.00 , 0.25)	48.4	0.9575 (± 0.0023)	46.7	0.9513 (± 0.0028)	46.9	0.9519 (± 0.0028)
50	(1.00 , 0.50)	50.6	0.9573 (± 0.0015)	48.8	0.9518 (± 0.0020)	49.2	0.9529 (± 0.0018)
50	(1.00 , 1.00)	57.0	0.9568 (± 0.0032)	56.9	0.9563 (± 0.0029)	56.8	0.9563 (± 0.0029)
50	(1.00 , 1.50)	64.5	0.9432 (± 0.0037)	68.8	0.9564 (± 0.0038)	69.1	0.9572 (± 0.0036)
50	(1.00 , 2.00)	70.5	0.9224 (± 0.0032)	83.2	0.9561 (± 0.0047)	81.1	0.9512 (± 0.0037)
50	(1.00 , 3.00)	77.8	0.8562 (± 0.0026)	110.1	0.9502 (± 0.0016)	113.9	0.9566 (± 0.0018)
50	(2.00 , 0.25)	27.8	0.9676 (± 0.0028)	24.1	0.9511 (± 0.0032)	24.3	0.9517 (± 0.0032)
50	(2.00 , 0.50)	28.6	0.9638 (± 0.0027)	25.7	0.9513 (± 0.0031)	25.3	0.9496 (± 0.0035)
50	(2.00 , 1.00)	32.1	0.9563 (± 0.0046)	31.9	0.9553 (± 0.0048)	31.8	0.9548 (± 0.0050)
50	(2.00 , 1.50)	36.2	0.9382 (± 0.0059)	40.6	0.9552 (± 0.0051)	40.2	0.9535 (± 0.0049)
50	(2.00 , 2.00)	38.8	0.9042 (± 0.0074)	49.7	0.9485 (± 0.0028)	48.4	0.9450 (± 0.0032)
50	(2.00 , 3.00)	41.5	0.8264 (± 0.0037)	53.7	0.8988 (± 0.0028)	68.2	0.9503 (± 0.0028)
50	(10.00 , 0.25)	10.0	0.9824 (± 0.0010)	6.6	0.9674 (± 0.0018)	4.8	0.9547 (± 0.0016)
50	(10.00 , 0.50)	10.0	0.9751 (± 0.0016)	7.6	0.9624 (± 0.0013)	6.5	0.9551 (± 0.0014)
50	(10.00 , 1.00)	10.6	0.9589 (± 0.0023)	10.3	0.9574 (± 0.0024)	9.5	0.9531 (± 0.0025)
50	(10.00 , 1.50)	11.4	0.9374 (± 0.0024)	9.4	0.9219 (± 0.0018)	13.7	0.9514 (± 0.0026)
50	(10.00 , 2.00)	11.8	0.9164 (± 0.0034)	-36.1	0.0995 (± 0.0021)	16.9	0.9508 (± 0.0032)
50	(10.00 , 3.00)	12.0	0.8867 (± 0.0028)	-3.3	0.6481 (± 0.0042)	21.2	0.9532 (± 0.0017)
100	(0.50 , 0.25)	81.5	0.9559 (± 0.0024)	80.1	0.9515 (± 0.0022)	81.0	0.9541 (± 0.0022)
100	(0.50 , 0.50)	83.4	0.9563 (± 0.0022)	82.3	0.9530 (± 0.0019)	83.0	0.9552 (± 0.0020)
100	(0.50 , 1.00)	92.2	0.9557 (± 0.0016)	91.0	0.9532 (± 0.0018)	90.5	0.9522 (± 0.0016)
100	(0.50 , 1.50)	102.2	0.9514 (± 0.0042)	104.6	0.9554 (± 0.0040)	104.3	0.9551 (± 0.0041)
100	(0.50 , 2.00)	112.4	0.9398 (± 0.0033)	121.9	0.9551 (± 0.0019)	119.4	0.9510 (± 0.0020)
100	(0.50 , 3.00)	126.9	0.8974 (± 0.0020)	162.7	0.9554 (± 0.0018)	158.6	0.9505 (± 0.0018)
100	(1.00 , 0.25)	39.3	0.9557 (± 0.0025)	38.0	0.9510 (± 0.0019)	37.7	0.9499 (± 0.0022)
100	(1.00 , 0.50)	40.9	0.9536 (± 0.0012)	39.6	0.9500 (± 0.0007)	39.0	0.9478 (± 0.0010)
100	(1.00 , 1.00)	46.0	0.9543 (± 0.0037)	45.9	0.9540 (± 0.0036)	44.3	0.9496 (± 0.0051)
100	(1.00 , 1.50)	52.0	0.9452 (± 0.0039)	55.7	0.9538 (± 0.0026)	54.9	0.9520 (± 0.0027)
100	(1.00 , 2.00)	57.0	0.9262 (± 0.0042)	68.0	0.9534 (± 0.0042)	66.8	0.9502 (± 0.0040)
100	(1.00 , 3.00)	63.2	0.8717 (± 0.0023)	92.8	0.9478 (± 0.0011)	94.3	0.9503 (± 0.0013)
100	(2.00 , 0.25)	20.0	0.9619 (± 0.0026)	17.4	0.9504 (± 0.0024)	18.0	0.9527 (± 0.0027)
100	(2.00 , 0.50)	20.6	0.9597 (± 0.0032)	18.5	0.9518 (± 0.0025)	18.4	0.9508 (± 0.0029)
100	(2.00 , 1.00)	23.0	0.9549 (± 0.0038)	23.0	0.9547 (± 0.0038)	22.3	0.9513 (± 0.0037)
100	(2.00 , 1.50)	26.2	0.9367 (± 0.0068)	29.7	0.9486 (± 0.0067)	27.6	0.9412 (± 0.0069)
100	(2.00 , 2.00)	28.2	0.9080 (± 0.0064)	37.1	0.9454 (± 0.0041)	36.1	0.9409 (± 0.0053)
100	(2.00 , 3.00)	30.3	0.8547 (± 0.0049)	40.6	0.9039 (± 0.0047)	54.4	0.9479 (± 0.0029)